SOLUTION OF THE MIXED PROBLEM FOR THIN BODIES ON AN ELASTIC FOUNDATION

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Solution of the mixed static problem for an infinite string, membrane, and plate partially supported on an elastic foundation is examined in this paper. The mixed nature of the problem is that a load is given on one part of the body, and on the other is equal to the reaction of the elastic foundation, which is unknown prior to solution of the problem. This reaction is proportional to the normal displacement of the body (Winkler–Voss hypothesis).

A method is used below which had been used in [1] to solve the elastic problem of a half space partially being supported on an elastic foundation.

We consider the external load applied in a bounded domain V, and denote the domain lying on the elastic foundation by P.

1. STRING ON AN ELASTIC FOUNDATION

Let the string be along the OX axis, the domain V occupies the segment $|x| < a$. The string displacements w satisfy the differential equations

$$d^2w/dx^2 = p(x),\ x \in V; \tag{1.1}$$

$$d^2w/dx^2 = \lambda w(x),\ x \in P, \tag{1.2}$$

where p(x) is a given total function of the load in the domain V, $\lambda > 0$ is the stiffness of the elastic foundation (bedding coefficient). A condition at infinity $w(x) \to 0$ as $|x| \to \infty$ and the condition of conjugation of the solution at the points $x = \pm a$

$$w_+(\pm a) = w_-(\pm a),\ (dw/dx)_+(\pm a) = (dw/dx)_-(\pm a), \tag{1.3}$$

must be appended to (1.1) and (1.2), where the plus (+) and minus (–) denote the limit values upon approaching the point $x = \pm a$ from the domains P and V, respectively.

Let us reduce problem (1.1)-(1.3) to an integral equation by using a specially selected fundamental solution. The fundamental solution with as yet unknown distribution density $\beta(\xi)$ is sought as the solution of the equation

$$d^2w/dx^2 = \lambda w(x) + \beta(\xi)\delta(x - \xi),\ -\infty < x < \infty,\quad w(x) \to 0 \text{ for } |x| \to \infty, \tag{1.4}$$

where $\delta(x - \xi)$ is the Dirac delta function. The physical meaning of (1.4) is that the deflection is sought for an infinite string lying on an elastic foundation and loaded at the point $x = \xi$ by a lumped force $\beta(\xi)$. The solution of problem (1.4) is obtained easily by using the Fourier transformation

$$w(x) = -\frac{\beta(\xi)}{2\pi}\int_{-\infty}^{\infty}\frac{e^{iq(x-\xi)}dq}{\lambda + q^2} = -\frac{\beta(\xi)\,e^{-\sqrt{\lambda}|x-\xi|}}{2\sqrt{\lambda}} = -\frac{\beta(\xi)\,G(x-\xi)}{\lambda}. \tag{1.5}$$

On the basis of the superposition principle, solution (1.5) integrated with respect to ξ in the domain V will satisfy (1.2) for $x \in P$. By satisfying (1.1) we obtain an equation in the density β

$$\beta(x) = p(x) + \int_V \beta(\xi)\,G(x - \xi)\,d\xi,\qquad x \in V. \tag{1.6}$$

The physical meaning of the function G is that it is the reaction of an infinite string on an elastic foundation to a unit lumped force. After having determined the function β from (1.6), the string deflections are determined from the formula

$$w(x) = -\frac{1}{\lambda}\int_V \beta(\xi)\,G(x - \xi)\,d\xi,\quad -\infty < x < \infty. \tag{1.7}$$

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Formulas (1.6) and (1.7) remain valid even in the case when V is an arbitrary bounded domain. It follows from the representation (1.7) that w(x) satisfies the conjugation conditions (1.3). If $x \in V$, then on the basis of (1.6) and (1.7) it is more convenient to calculate the deflection w from the relationship

$$p(x) = \beta(x) + \lambda w(x). \tag{1.8}$$

Let us examine the properties of the integral operator K in the right side of (1.6) in more detail. The kernel G of this integral operator is positive and $\int\limits_{-\infty}^{\infty} G(t)\,dt = 1$. The physical meaning of this last equality is that the reaction of elastic framing equals the applied external force. Using the properties of the kernel noted and considering the domain V bounded, we obtain [2]

$$\| K \| = \max_{x \in V} \int\limits_{V} G(x-\xi)\,d\xi < \int\limits_{-\infty}^{\infty} G(t)\,dt = 1, \tag{1.9}$$

where here and later $\|K\|$ denotes the norm of the operator K acting in the space of summable functions $L_1(V)$ or of continuous functions C(V) in the closure V. If $p \in L_1(V) (p \in C(V))$, then the inequality (1.9) permits representation of the solution of (1.6) in the form of a Neumann series that converges normally in $L_1(V)$ [in C(V)], and assertion on the basis of the principle of compressed mappings [2] that the solution obtained is unique in $L_1(V)$ and C(V). Condition (1.9) also permits finding the solution (1.6) by using successive approximations. The approximations obtained will here be partial sums of the Neumann series. The possibility of using successive approximations is quite convenient in the calculational plan for the case when the domain V is not simply connected.

Let us consider certain properties of the solution of the problem (1.1)-(1.3).

A. If $p(x) > 0$ for $x \in V$, then $\beta(x) > 0$ and $w(x) < 0$ on the basis of (1.8) in the domain P, and tends monotonically to zero as $|x| \to \infty$. This follows easily from the properties of the kernel and representation (1.6)-(1.8).

B. Let the reaction of the elastic foundation in the domain P be denoted analogously to the external force $p(x) = \lambda w(x),\ x \in P$. Let the load $p(x),\ x \in V,$ be such that there exists $p_+(s) = \lim\limits_{x \to s} p(x),\ x \in V$, where s is a point lying on the interface Γ of the domains P and V. By virtue of the properties of the solution obtained there exists $p_-(s) = \lim\limits_{x \to s} p(x), x \in V, s \in \Gamma$. Then the formula for the jump follows from (1.6) and (1.7)

$$p_+(s) - p_-(s) = \beta(s), \quad \beta(s) = \lim_{x \to s} \beta(x), \quad x \in V, \quad s \in \Gamma. \tag{1.10}$$

The relationship (1.10) is the result of (1.8) and the first condition in (1.3). The most important result of the jump formula is the continuity condition for the function p(x) at the points $\pm a$: $\beta(s) = 0$, $s = \pm a$.

The properties of the solution are analogous to those obtained in [1]. However, the possibility of obtaining a solution in the final expressions for the string permits reformulation of the continuity condition for p(x) during passage through the boundary Γ in quantities known a priori. We use the notation

$$q(x) = \int\limits_{-a}^{x} \int\limits_{-a}^{\eta} p(\xi)\,d\xi\,d\eta \qquad \int\limits_{-a}^{x} p(\xi)(x-\xi)\,d\xi,$$

$$A_1 = -\frac{q'(a) + \sqrt{\lambda}\,q(a)}{2(1 + a\sqrt{\lambda})}, \quad A_0 = -\frac{q'(a) + \sqrt{\lambda}\,q(a)}{2\sqrt{\lambda}}$$

(the primes denote differentiation). By definition of p(x) we have in the domain P

$$p(x) = \lambda w(x) = -\lambda(q(x) + A_1 x + A_0)$$

and the continuity condition for p(x) takes the form

$$p_+(s) - \lambda(q(s) + A_1 s + A_0) = 0, \ s = \pm a. \tag{1.11}$$

Condition (1.11) can be simplified by using the symmetry of p(x). In the case of an even (odd) function p(x) the continuity condition has the form

$$p_+(s) - \lambda \int_{-a}^{s} p(\xi)(s-\xi)\,d\xi + \frac{\sqrt{\lambda}(1+\sqrt{\lambda}(s+a))}{2} \int_{-a}^{a} p(\xi)\,d\xi = 0$$

$$\left(p_+(s) - \lambda \int_{-a}^{s} p(\xi)(s-\xi)\,d\xi - \frac{\lambda(1+\sqrt{\lambda}(s+a))}{2(1+a\sqrt{\lambda})} \int_{-a}^{a} \xi p(\xi)\,d\xi = 0\right), \quad s = \pm a.$$

The equality (1.11) imposes a constraint on the known function p(x) from (1.1).

2. MEMBRANE ON AN ELASTIC FOUNDATION

Let the membrane occupy the plane OXY. In this case we have instead of (1.1)-(1.3)

$$\Delta w = p(x, y), \ (x, y) \in V; \tag{2.1}$$

$$\Delta w = \lambda w, \ (x, y) \in P, \tag{2.2}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator. The condition at infinity

$$w(x, y) \to 0 \ \text{for} \ r = \sqrt{x^2+y^2} \to \infty \tag{2.3}$$

and the condition on the interface Γ between domains P and V

$$w_+ = w_-, \ (\partial w/\partial n)_+ = (\partial w/\partial n)_- \tag{2.4}$$

must be appended to (2.1) and (2.2). The subscripts +, — have the same meaning as in Sec. 1, and $\partial/\partial n$ is the derivative along the normal to Γ.

We use the solution of the auxiliary problem

$$\Delta w = \lambda w(x, y) + \beta(\xi, \eta)\delta(x-\xi, y-\eta), \ r \geqslant 0, \quad w \to 0, \ r \to \infty, \tag{2.5}$$

to solve the formulated problem (2.1)-(2.4), where $\delta(x-\xi, y-\eta)$ is the Dirac delta-function The physical meaning of problem (2.5) is that the deflection is sought for a membrane occupying the whole OXY plane and lying completely on an elastic foundation loaded here at the point (ξ, η) by a lumped force $\beta(\xi, \eta)$.

Because of axial symmetry the solution of problem (2.5) is easily obtained by using the Hankel transform:

$$w(x, y) = -\frac{\beta(\xi, \eta)}{2\pi} \int_0^\infty \frac{tJ_0(\rho t)\,dt}{\lambda + t^2} = -\frac{\beta(\xi, \eta)}{2\pi} K_0(\rho\sqrt{\lambda}) = -\frac{\beta(\xi, \eta) G(\rho)}{\lambda}, \quad \rho = \sqrt{(x-\xi)^2 + (y-\eta)^2}, \tag{2.6}$$

where

$$K_0(x) = -\ln\frac{x}{2} I_0(x) + \sum_{i=0}^{\infty} \left(\frac{x^i}{2^i i!}\right)^2 \psi(i+1) \tag{2.7}$$

is the Macdonald function [3], and $\psi(i+1) = -0.577215 + \sum_{n=1}^{i} \frac{1}{n}$ is the Euler ψ-function. The physical meaning of G is the same as in the problem of a string (Sec. 1).

Let $(\xi, \eta) \in V$, then if $\beta(\xi, \eta) = 0$ for $(\xi, \eta) \in P$, the fundamental solution (2.7) satisfies Eq. (2.2) in the domain P. We take the superposition of the fundamental solutions (2.6) in such a manner as to satisfy (2.1). We obtain an integral eqution in the density β

$$\beta(x, y) - \iint_V \beta(\xi, \eta) G(\rho)\, d\xi d\eta = p(x, y), \quad (x, y) \in V \tag{2.8}$$

and an integral representation for the solution of problem (2.1)-(2.4)

$$w(x, y) = -\frac{1}{\lambda} \iint_V \beta(\xi, \eta) G(\rho)\, d\xi d\eta, \quad -\infty < x, y < \infty. \tag{2.9}$$

Let us note as a result of (2.8) and (2.9) a formula representing the dependence between the distribution density for the lumped forces β, the displacement w, and a given load p in the domain V:

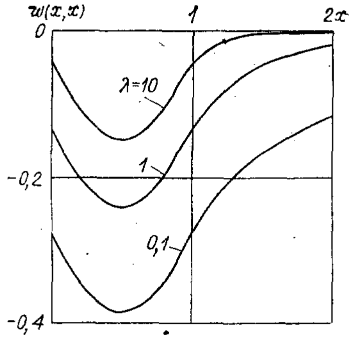

Fig. 1

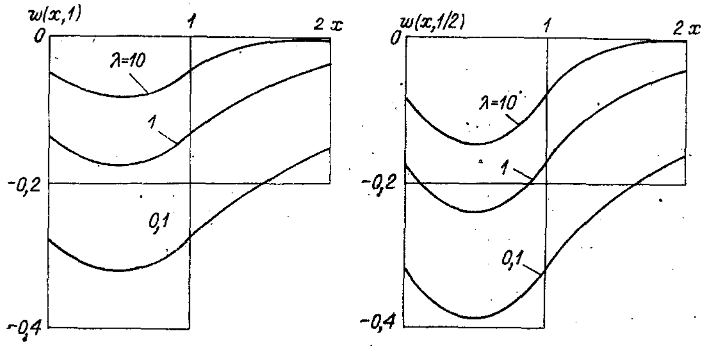

Fig. 2

$$p(x, y) = \beta(x, y) + \lambda w(x, y),\ (x, y) \in V. \tag{2.10}$$

Let us list the properties of the kernel G that are needed later:

$$G(\rho) > 0,\quad \iint_{-\infty}^{\infty} G(\rho)\, d\xi d\eta = 1.$$

To prove the last equality, it is sufficient to go over to polar coordinates and to use formula 6.521.2 [3]. The mechanical meaning of this relationship is the same as in the case of the string. In contrast to the case of the string, the kernel G has a logarithmic singularity at zero [see (2.7)].

These properties of the kernel permit literal repetition of the derivation of almost all the properties of the solution analogously to the case of the string in Sec. 1, starting with (1.9). The distinction is just that the continuity condition for p(x, y) for passage through the boundary Γ is not successfully written down explicitly.

Because (2.8) allows of solution by successive approximations in the case when the domain V is bounded, the numerical solution of problem (2.1)-(2.4) is possible for a sufficiently complex configuration of the domain V. Therefore, the solution of (2.8) reduces, in practice, to a calculation for the (n + 1)-th iteration of expressions of the form $\iint_V \beta_n(\xi, \eta)\, G(\rho)\, d\xi d\eta + p(x, y)$. After having solved (2.8) with adequate accuracy, the membrane deflections w are determined by means of (2.9) [it is more convenient to use (2.10) for the domain V].

To illustrate the possibilities of the method, the problem (2.1)-(2.4) was solved numerically in the case when the domain V is a square $0 < x,\ y < 1$. The load p(x, y) in the domain V was assumed constant (it can be considered one without limiting the generality).

Diagrams of the membrane deflections along the line $y = x$ $(x \geqslant 0)$ are shown in Fig. 1, and along the lines $y = 1$, $y = 1/2$ $(x \geqslant 0)$ in Fig. 2. It is characteristic for the deflection distribution that upon reaching the extremum for $x = 1/2$, $y = 1/2$ they grow monotonically to zero at infinity. By using Figs. 1 and 2 it is easy to obtain the reaction of the elastic foundation at points where the deflections are presented. To do this it is sufficient to multiply the value of the deflection by the stiffness λ of the elastic foundation.

3. PLATE ON AN ELASTIC FOUNDATION

Retaining the notation and assumptions used in the membrane problem (Sec. 2), we examine the problem of a plate lying partially on a Winkler elastic foundation. We have

$$\Delta^2 w = -p(x, y),\ (x, y) \in V; \tag{3.1}$$

$$\Delta^2 w = -\lambda w,\ (x, y) \in P, \tag{3.2}$$

where $\Delta^2 = \partial^2/\partial x^4 + 2\partial^4/\partial x^2 \partial y^2 + \partial^4/\partial y^4$ is the biharmonic operator, the condition at infinity is $w(x, y) \to 0$ as $r \to \infty$ and the condition on the interface Γ of the domains P and V is

$$w_+ = w_-,\ (\partial^k w/\partial n^k)_+ = (\partial^k w/\partial n^k)_-,\ k = 1, 2, 3. \tag{3.3}$$

Analogously to Secs. 1 and 2, we consider the solution of the auxiliary problem

$$\Delta^2 w = -\lambda w(x, y) + \beta(\xi, \eta)\delta(x - \xi, y - \eta),\ r \geqslant 0,$$
$$w \to 0,\ r \to \infty, \tag{3.4}$$

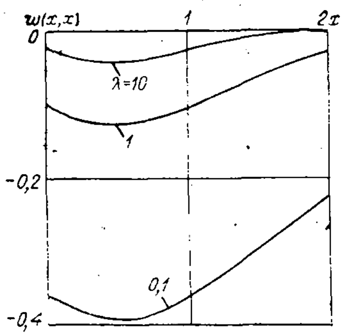

Fig. 3

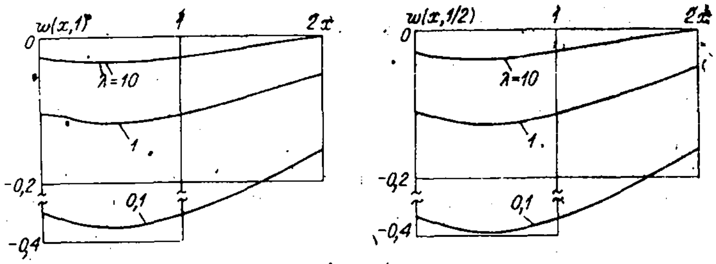

Fig. 4

which can be obtained by using the Hankel transform

$$w(x, y) = +\frac{\beta(\xi, \eta)}{2\pi} \int_0^\infty \frac{tJ_0(\rho t)\, dt}{\lambda + t^4} = \frac{-\beta(\xi, \eta)}{2\pi\sqrt{\lambda}} \operatorname{kei}\left(\rho \sqrt[4]{\lambda}\right) = +\frac{\beta(\xi, \eta)\, G(\rho)}{\lambda}$$

(kei (x) is the Thomson function [3]).

Let us suppose the fundamental solution by considering that $(\xi, \eta) \in V$, $\beta(\xi, \eta) = 0$ for $(\xi, \eta) \in P$. Then Eq. (3.2) and condition (3.3) at infinity are satisfied because of the properties of the fundamental solution. Satisfying (3.1) we obtain an integral equation in the density β

$$\beta(x, y) - \iint_V \beta(\xi, \eta)\, G(\rho)\, d\xi d\eta = -p(x, y), \quad (x, y) \in V. \tag{3.5}$$

Superposition of the fundamental solution in the domain V yields an integral representation for the solution of the problem (3.1)-(3.3) in the whole plane:

$$w(x, y) = +\frac{1}{\lambda} \iint_V \beta(\xi, \eta)\, G(\rho)\, d\xi d\eta, \quad -\infty < x, y < \infty. \tag{3.6}$$

Analogously to the membrane problem, we have for the domain V the relationship

$$-p(x, y) = \beta(x, y) - \lambda w(x, y), \ (x, y) \in V.$$

Therefore, the problem of seeking the solution of (3.1), (3.2) is reduced to the integral equation (3.5) in the auxiliary function β. After this latter has been solved, the solution of the initial problem is determined by the representation (3.6).

In contrast to Secs. 2 and 1, the kernel G of (3.5) is a smooth sign-variable function, where a neighborhood of zero exists in which G > 0. As before, $\iint_{-\infty}^{\infty} G(\rho)\, d\xi d\eta = 1$ but it is impossible to derive the inequality $\|K\| < 1$ which underlies the investigations in Secs. 2 and 3. This is related to the sign-variability of the kernel.

However, for a sufficiently small domain V the inequality $\|K\| < 1$ remains valid. In fact, let ρ_0 denote a number such that

$$-\operatorname{kei}(\xi) \geqslant 0 \ \text{for} \ \xi < \rho_0, \quad -\int_0^{\rho_0} \xi \operatorname{kei}(\xi)\, d\xi = 1.$$

Let us select the diameter of the domain V as less than $(1/2)\rho_0\lambda^{-1/4}$. Then

$$\|K\| = \max_{(x,y)\in V} \iint_V |G(\rho)|\, d\xi d\eta < \iint_{\{s^2+t^2\}<\rho_0^2/\sqrt{\lambda}} G(\omega)\, ds dt = -\int_0^{2\pi} \frac{d\varphi}{2\pi} \int_0^{\rho_0} \xi \operatorname{kei}(\xi)\, d\xi = 1,$$

$$\omega = \sqrt{s^2 + t^2}.$$

For such domains V the results of Secs. 2 and 1 remain valid.

When the domain V is the square $0 < x, y < 1$ the problem was numerically solved by the method presented above for the case of a constant load in the domain V. The integral equation

(3.5) was solved by successive approximations here. The results of the numerical computations are presented in Figs. 3 and 4. The distribution of the function w along the line $y = x$ ($x \geqslant 0$) is represented in Fig. 3, and along the lines $y = 1$ and $y = 1/2$ ($x \geqslant 0$) in Fig. 4.

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BIPERIODIC SYSTEM OF RECTILINEAR LONGITUDINAL-SHEAR CRACKS IN AN ELASTIC BODY

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Problems of the theory of elasticity for an infinite isotropic body weakened by a biperiodic system of rectilinear cracks were examined in [1-11], where they were reduced to a numerical solution of a singular integral equation or an infinite algebraic system. In this article we construct an analytic solution to a problem for a biperiodic system of rectilinear longitudinal-shear cracks forming a rhombic network. An expression is obtained for the macroscopic shear modulus of a medium with such a system of cracks.

1. Formulation and Solution of the Biperiodic Problem. It is known [12] that the solutions of problems of longitudinal shear reduce to determination of the function F(z) analytic in the region occupied by the body, where $z = x + iy$. Here, the stress components σ_{xz} and σ_{yz} and the displacement w are determined from the formulas

$$\sigma_{xz} - i\sigma_{yz} = \mu_0 F(z),\ w = \operatorname{Re} f(z),\ F(z) = f'(z), \tag{1.1}$$

where μ_0 is the shear modulus.

Let an infinite elastic plane xOy be weakened by a biperiodic system of rectilinear slits parallel to the real axis. It is assumed that the fundamental parallelogram of periods has the form of a rhombus. A slit is located inside the parallelogram across the diagonal (Fig. 1). On the edges of the slits we specify a self-balanced load which is equal at congruent points

$$\sigma_{yz} = -T(x),\ |x| < l,\ y = 0. \tag{1.2}$$

We use 2g(x) to designate the discontinuity of the displacement in the transition across the slit

$$2g(x) = w(x, +0) - w(x, -0),\ |x| \leqslant l.$$

Let the applied load T(x) be an even function of the coordinate x. Then $T(x) = T(-x)$ and, by virtue of the symmetry of the problem, the function F(z) is an even biperiodic function. It can be shown [13, 14] that F(z) is expressed through the derivative of the function g(x) in the form

$$F(z) = \frac{1}{\pi i} \int_0^l \frac{g'(t) P'(t)\, dt}{P(t) - P(z)}, \tag{1.3}$$

where P(z) is an elliptic Weierstrass function. The primes denote differentiation with respect to the argument.

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